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Decay of Correlations for Piecewise Expanding Maps

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This paper investigates the decay of correlations in a large class of non-Markov one-dimensional expanding maps. The method employed is a special version of a general approach recently proposed by the author. Explicit bounds on the rate of decay of correlations are obtained.

KEY WORDS: Correlations; one-dimensional maps; transfer operator; Hilbert metric; exponential mixing; invariant measures.

INTRODUCTION

In this paper we apply a new technique—introduced in ref. 15 to study the decay of correlations in hyperbolic systems—to one-dimensional non-Markov expansive maps (see Section 2 for a precise definition). A main feature of such a method is the possibility of obtaining explicit bounds on the rate of decay.

The standard approach to this problem[2] is to study the Perron–Frobenius (PF) operator via Ionescu-Tulcea and Marinescu-type spectral theorems.[(14)] Unfortunately, this suffices to prove exponential decay of correlations, but does not provide any constructive bound on the rate of decay. One can hope to obtain some bounds using the theory of Ruelle zeta functions,[(17–19)] but it is not clear of the generality in which this can be accomplished.

Here, we will study the PF operator as well, but from a different point of view: we will see that there exists a convex cone of functions that is mapped "strictly" inside itself by the PF operator. We will then be able to

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[2] See ref. 16, which proves the existence of an invariant measure absolutely continuous with respect to Lebesgue, and ref. 11, where the decay of correlations is investigated.

take advantage of an idea by Garrett Birkhoff: he showed that one can associate a Hilbert metric to the above-mentioned cone and that such a metric is contracted by the PF operator. Once we obtain a contraction all the wanted consequences are the result of straightforward arguments. This will allow us to obtain an explicit bound for the rate of decay.

While several bounds on the rate of decay are available for smooth (or, more generally, Markov) expanding maps,[7, 20, 12] very little is known in the discontinuous case (more precisely, in the non-Markov case).

The possibility to obtain bounds on the rate of decay for non-Markov maps was recently investigated by Collet.[6] He used techniques developed for the study of Markov chains.[5] Here we show that the approach proposed in ref. 15 is more direct and produces substantially better bounds for the class of maps studied in ref. 6, and we apply it to a larger class of maps.

If T is a map of the type under consideration (see Section 2) and μ is the associated invariant probability measure, absolutely continuous with respect to the Lebesgue measure (i.e., there exists $\phi \in L^1([0, 1])$ such that $d\mu = \phi\, dx$), then I prove the following:

Theorem 0.1. If $\inf_{x \in [0,1]} \phi(x) \geqslant \gamma > 0$ and the system (T, μ) is mixing, then there exists b, $K > 0$, $\Lambda \in (0, 1)$, "constructive in T," such that, for each function $f \in L^1([0, 1])$ and $g \in BV$ (the space of functions of bounded variation), $\int_0^1 g = 1$,

$$\left| \int_0^1 g f \circ T^n - \int_0^1 f\, d\mu \right| \leqslant K\Lambda^{-n} \|f\|_1 \left(1 + b \bigvee_0^1 g \right)$$

In addition, it is possible to state a concrete and verifiable condition that ensures $\inf \phi \geqslant \gamma > 0$ and shows that it is a rather general feature of expanding maps (see Lemma 4.2). Of course, the novelty of Theorem 0.1 consists in "constructive in T," which means that, given T, it is possible to state explicit, although involved, formulas for b, K, and Λ. Such formulas are summarized for the reader's convenience in Appendix A. I do not claim that the explicit b, K, and Λ that I produce here are optimal (in fact, I believe they are not); more work in this direction is called for. Lastly, it may be interesting to notice that the bounds are robust: they vary continuously in the C^2 topology.

The paper is organized as follows: In Section 1, I provide a brief review of some basic facts on which the subsequent arguments rest. Section 2 contains a detailed description of the class of maps under consideration and it presents the main ingredient in the proof of Theorem 0.1: a cone of functions invariant under the action of the Perron–Frobenius operator.

In Section 3 the bound on the decay of correlations is proven, but under apparently stronger hypotheses than the ones in the statement of Theorem 0.1. In Section 4, I show that the hypotheses of Theorem 0.1 imply the assumptions used in Section 3, thereby concluding the argument. As a byproduct, I also answer some questions concerning general properties of expanding maps recently raised by Collet.(6) Finally, the paper includes two appendices; in the first I summarize the formulas for the constants b, K, and Λ and discuss a concrete example, obtaining numerical values for such constants (thereby emphasizing the constructive nature of the present approach). In the second I prove a helpful inequality.[3]

1. OPERATORS AND INVARIANT CONVEX CONES

This section illustrates some results in lattice theory originally due to Birkhoff.(4) More details and the proofs of the following results can be found in ref. 15.

Consider a topological vector space $\mathbb{V}$, with a partial ordering "$\preccurlyeq$," that is a vector lattice.[4] We require the partial order to be "continuous," i.e., given $\{f_n\} \in \mathbb{V}$, $\lim_{n \to \infty} f_n = f$, if $f_n \succcurlyeq g$ for each n, then $f \succcurlyeq g$. We call such vector lattices "integrally closed."[5]

We define the closed convex cone[6] $\mathscr{C} = \{f \in \mathbb{V} \mid f \neq 0,\ f \succcurlyeq 0\}$ (hereafter, the term "closed cone" $\mathscr{C}$ will mean that $\mathscr{C} \cup \{0\}$ is closed), and the equivalence relation "$\sim$": $f \sim g$ iff there exists $\lambda \in \mathbb{R}^+ \setminus \{0\}$ such that $f = \lambda g$. If we call $\tilde{\mathscr{C}}$ the quotient of $\mathscr{C}$ with respect to $\sim$, then $\tilde{\mathscr{C}}$ is a closed convex set. Conversely, given a closed convex cone $\mathscr{C} \subset \mathbb{V}$ enjoying the property $\mathscr{C} \cap -\mathscr{C} = \varnothing$, we can define an order relation by

$$f \preccurlyeq g \Leftrightarrow g - f \in \mathscr{C} \cup \{0\}$$

Henceforth, each time that we specify a convex cone we will assume the corresponding order relation and vice versa.

[3] The inequality is far from new. I prove it for the convenience of the reader, because the form I use is slightly nonstandard.

[4] We are assuming the partial order to be well behaved with respect to the algebraic structure: for each $f, g \in \mathbb{V}$, $f \succcurlyeq g \Leftrightarrow f - g \succcurlyeq 0$; for each $f \in \mathbb{V}$, $\lambda \in \mathbb{R}^+ \setminus \{0\}$, $f \succcurlyeq 0 \Rightarrow \lambda f \succcurlyeq 0$; for each $f \in \mathbb{V}$, $f \succcurlyeq 0$ and $f \preccurlyeq 0$ imply $f = 0$ (antisymmetry of the order relation).

[5] To be precise, in the literature "integrally closed" is used in a weaker sense. First, $\mathbb{V}$ does not need a topology. Second, it suffices that for $\{\alpha_n\} \in \mathbb{R}$, $\alpha_n \to \alpha$; $f, g \in \mathbb{V}$, if $\alpha_n f \succcurlyeq g$, then $\alpha f \succcurlyeq g$. Here we will ignore these and other subtleties: our task is limited to a brief account of the results relevant to the present context.

[6] Here, by "cone" we mean any set such that, if f belongs to the set, then λf belongs to it as well, for each $\lambda > 0$.

It is then possible to define a projective metric Θ (Hilbert metric)[7] in $\mathscr{C}$ by the construction

$$\alpha(f, g) = \sup\{\lambda \in \mathbb{R}^+ \mid \lambda f \preccurlyeq g\}$$

$$\beta(f, g) = \inf\{\mu \in \mathbb{R}^+ \mid g \preccurlyeq \mu f\}$$

$$\Theta(f, g) = \log\left[\frac{\beta(f, g)}{\alpha(f, g)}\right]$$

where we take $\alpha = 0$ and $\beta = 0$ if the corresponding sets are empty.

The importance of the previous constructions is due, in our context, to the following theorem.

Theorem 1.1. Let $\mathbb{V}_1$ and $\mathbb{V}_2$ be two integrally closed vector lattices, and $T: \mathbb{V}_1 \to \mathbb{V}_2$ a linear map such that $T(\mathscr{C}_1) \subset \mathscr{C}_2$ for the two corresponding closed convex cones $\mathscr{C}_1 \subset \mathbb{V}_1$ and $\mathscr{C}_2 \subset \mathbb{V}_2$. Let Θ_i be the Hilbert metric corresponding to the cone $\mathscr{C}_i$. Setting $\Delta = \sup_{f, g \in T(\mathscr{C}_1)} \Theta_2(f, g)$, we have

$$\Theta_2(Tf, Tg) \leqslant \tanh\left(\frac{\Delta}{4}\right) \Theta_1(f, g), \qquad \forall f, g \in \mathscr{C}_1$$

$[\tanh(\infty) \equiv 1]$.

Remark 1.2. If $T(\mathscr{C}_1) \subset \mathscr{C}_2$, then it follows that $\Theta_2(Tf, Tg) \leqslant \Theta_1(f, g)$. However, a uniform rate of contraction depends on the diameter of the image being finite.

In particular, if an operator maps a convex cone strictly inside itself (in the sense that the diameter of the image is finite), then it is a contraction in the Hilbert metric. This implies the existence of a "positive" eigenfunction (provided the cone is complete with respect to the Hilbert metric), and, with some additional work, the existence of a gap in the spectrum of T (see ref. 3 for details). The relevance of this theorem for the study of invariant measures and their ergodic properties is obvious.

It is natural to wonder about the strength of the Hilbert metric compared to other, more usual, metrics. While in general the answer depends on the cone, it is nevertheless possible to state an interesting result.

[7] In fact, we define a semimetric, since $f \sim g \Rightarrow \Theta(f, g) = 0$. The metric that we describe corresponds to the conventional Hilbert metric on $\tilde{\mathscr{C}}$.

Lemma 1.3. Let $\|\cdot\|$ be a norm on the vector lattice $\mathbb{V}$, and suppose that, for each f, $g \in \mathbb{V}$,

$$-f \preccurlyeq g \preccurlyeq f \Rightarrow \|f\| \geqslant \|g\|$$

Then, given f, $g \in \mathscr{C} \subset \mathbb{V}$ for which $\|f\| = \|g\|$,

$$\|f - g\| \leqslant (e^{\Theta(f, g)} - 1)\, \|f\|$$

2. THE MAP AND AN INVARIANT CONE

We consider a map T from the interval $[0, 1]$ into itself.

We assume that there exists a finite partition (mod 0, with respect to the Lebesgue measure) $\mathscr{A}_0$ of $[0, 1]$ into open intervals such that, for each interval $I \in \mathscr{A}_0$, the map T, restricted to I, can be extended to a C^2 map on an open interval containing the closure of I. In addition, we assume that $|DT| \geqslant \lambda > 1$ (expansivity).[8]

The simplest examples in this class of maps occur when, for each $I \in \mathscr{A}_0$, TI is equal, mod 0, to the union of elements of $\mathscr{A}_0$. This case is called the "Markov case," and it is well understood in the literature; in particular, explicit bounds on the rate of decay of correlations are available.[7, 20, 12, 15] If the above-mentioned property fails, the map is called "non-Markov"; this is the case addressed here.

Thanks to the work of Lasota and Yorke,[16] it is known that any piecewise smooth expanding map has at least one invariant measure absolutely continuous with respect to Lebesgue.[9] We will call ϕ the density, with respect to the Lebesgue measure, of such an invariant measure μ, and assume that the dynamical system (T, μ) is mixing.

The results of refs. 1 and 11 imply that the class of maps under consideration exhibits exponential decay of correlations. But no explicit bound on the rate of decay is available in such a generality.

A first technical obstacle is that the set $\{x \in [0, 1] \mid \phi(x) = 0\}$ may have positive Lebesgue measure. This is a concrete possibility and it is very easy to construct examples with this property. The simplest case of this behavior occurs when there exists an attracting set; clearly such a set supports all the invariant measures in the class under consideration (i.e., measures that are absolutely continuous with respect to the Lebesgue

[8] The requirement $\mathscr{A}_0$ finite is not essential, but certainly several more extra conditions on the map should be introduced if $\mathscr{A}_0$ is countable. For example, the condition $\inf_{I \in \mathscr{A}_0} |TI| > 0$ is necessary for the main inequality (2.1) to make sense and to hold. Moreover, additional hypotheses would be needed to prove Lemma 3.5.

[9] See ref. 1 for a generalization of such a result.

measure). We will assume that this is not the case. In principle, such an assumption does not imply any loss of generality: we can always study the restriction of the map to an invariant set.[10] Yet, some more complex situations may be possible. For example, $\phi(x) \neq 0$ for each $x \in [1, 0]$, but $\inf_{x \in [0,1]} \phi(x) = 0$. As we will see in Lemma 4.1, $\inf \phi = 0$ implies that some sets are visited very seldom; this makes it much harder to estimate the rate of decay. To avoid such complications, let us further reduce the class of maps under consideration.

On one hand, we assume that $|DT| \geqslant \lambda > 2$ (here no loss of generality is implied: this condition can always be satisfied replacing T by an appropriate power). On the other hand, we ask that there exists $\gamma > 0$ such that $\inf_{x \in [0,1]} \phi(x) \geqslant \gamma$.

To study the decay of correlations it is useful to introduce the Perron–Frobenius (PF) operator $\tilde{T}$. Such an operator is defined by the relation[11]

$$\int gf \circ T = \int f \tilde{T} g$$

for each $f \in L^\infty([0, 1])$ and $g \in L^1([0, 1])$. A direct computation shows that

$$\tilde{T}g(x) = \sum_{y \in T^{-1}(x)} g(y) \, |D_y T|^{-1}$$

We will show that the PF operator leaves invariant a cone of functions; the decay of the correlations will then follow from the theory discussed in Section 1.

Let BV be the space of functions of bounded variation on $[0, 1]$. The cones that we will use in this paper are

$$\mathscr{C}_a = \left\{ g \in BV \;\middle|\; g(x) \not\equiv 0;\ g(x) \geqslant 0\ \forall x \in [0, 1];\ \bigvee_0^1 g \leqslant a \int_0^1 g \right\}$$

for $a > 0$.[12]

The main ingredient for studying the maps under consideration is the following inequality (due to Lasota and Yorke,$^{(16)}$ but see Appendix B for details): for each $g \in BV$,

$$\bigvee_0^1 \tilde{T} g \leqslant 2\lambda^{-1} \bigvee_0^1 g + A \int_0^1 |g| \tag{2.1}$$

[10] In fact, there is nothing sacred about the interval [0, 1].

[11] If not otherwise stated, all the integrals are between 0 and 1, and taken with respect to the Lebesgue measure.

[12] By $\bigvee_0^1 g$ we mean the variation of the function g in the interval [0, 1].

where

$$A = \sup_{\xi \in [0,1]} \frac{|D_\xi^2 T|}{|D_\xi T|^2} + 2 \sup_{I \in \mathscr{A}_0} \frac{\sup_{\xi \in I} |D_\xi T|^{-1}}{|I|}$$

($\mathscr{A}_0$ and λ are defined at the beginning of the section).[13] The relevance of the cones $\mathscr{C}_a$ and the inequality (2.1) is exemplified by the following result.

Lemma 2.1. For each $a > A/(1-2\lambda^{-1})$ there exists $\sigma < 1$ such that

$$\tilde{T}\mathscr{C}_a \subset \mathscr{C}_{\sigma a}$$

Proof. For each $g \in \mathscr{C}_a$, inequality (2.1) yields

$$\bigvee_0^1 \tilde{T}g \leqslant 2\lambda^{-1} \bigvee_0^1 g + A \int_0^1 g \leqslant (2\lambda^{-1}a + A) \int_0^1 g$$

The result follows by choosing $\sigma = 2\lambda^{-1} + Aa^{-1}$ and noticing that $\int g = \int \tilde{T}g$. ∎

3. DECAY OF CORRELATIONS

In the previous section we found a cone of functions that is left invariant by the PF operator. This it is not quite enough to obtain a contraction in the corresponding Hilbert metric: the diameter of the image must also be investigated. The nature of the problem is elucidated by the following lemma.

Lemma 3.1. Calling Θ_a the Hilbert metric associated to the cone $\mathscr{C}_a$, for each $\nu < 1$ and $g \in \mathscr{C}_{\nu a}$

$$\Theta_a(g, 1) \leqslant \ln \left[\frac{\max\{(1+\nu)\int_0^1 g; \sup_{x \in [0,1]} g(x)\}}{\min\{(1-\nu)\int_0^1 g; \inf_{x \in [0,1]} g(x)\}} \right]$$

[13] The alert reader has certainly noticed that we have defined $\tilde{T}$ only on $L^1([0,1])$; to define it on BV, it is necessary to specify the value of $\tilde{T}^n g$ at all points, that is, also at the boundaries of the intervals belonging to the partition $\mathscr{A}_0$. In fact, we want to define all the powers of $\tilde{T}$ as well, so we may be in trouble at countably many points (all the preimages of the boundaries of the partition). We will not worry about such points, since they can be consistently ignored (loosely speaking, the value of $\tilde{T}g$ at such points can be defined by taking left and/or right limits—see ref. 11 or ref. 1 for details).

Proof. We have to find the set of λ and μ such that $\lambda 1 \preccurlyeq g \preccurlyeq \mu 1$. Let us start with the first inequality: it is satisfied iff $g - \lambda \in \mathscr{C}_a$, i.e.,

$$\lambda \leqslant g(x) \qquad \forall x \in [0, 1]$$

$$\lambda \leqslant \int_0^1 g - a^{-1} \bigvee_0^1 g$$

Consequently,

$$\alpha = \sup \lambda = \min \left\{ \inf_{x \in [0, 1]} g(x); \int_0^1 g - a^{-1} \bigvee_0^1 g \right\}$$

See Section 1 for a definition of α and β in the present context. Since $g \in \mathscr{C}_{\nu a}$, it follows that

$$\int_0^1 g - a^{-1} \bigvee_0^1 g \geqslant (1 - \nu) \int_0^1 g$$

that is, $\alpha \geqslant \min\{(1 - \nu) \int g, \inf g\}$. Analogously, one can compute $\beta \leqslant \max\{(1 + \nu) \int g, \sup g\}$. ∎

Note that, up to now, we do not have any control on the inf of a function belonging to our cones; therefore the above lemma shows not only that more work is needed, but also in which direction to concentrate our efforts.

The first step is to notice that, if a function belongs to the cone $\mathscr{C}_a$, then it cannot be small too often. This is made precise by the following.

Lemma 3.2. Given a partition, mod 0, $\mathscr{P}$ of $[0, 1]$, if each $p \in \mathscr{P}$ is a connected interval with Lebesgue measure less than $1/(2a)$ [that is, $|p| < 1/(2a)$], then, for each $g \in \mathscr{C}_a$, there exists $p_0 \in \mathscr{P}$ such that

$$g(x) \geqslant \tfrac{1}{2} \int_0^1 g \qquad \forall x \in p_0$$

Proof. Consider the set

$$\mathscr{P}_- = \left\{ p \in \mathscr{P} \;\middle|\; \exists x_p \in p \colon g(x_p) < \frac{1}{2} \int_0^1 g \right\}$$

Clearly the lemma is proven if we show that $\mathscr{P}_- \neq \mathscr{P}$. Let us suppose the contrary. For

$$\int_p g \leqslant |p| \left(g(x_p) + \bigvee_p g \right) < \frac{|p|}{2} \int_0^1 g + \frac{1}{2a} \bigvee_p g$$

and remembering that $g \in \mathscr{C}_a$, we have

$$\int_0^1 g < \frac{1}{2}\int_0^1 g + \frac{1}{2a}\bigvee_0^1 g \leqslant \int_0^1 g$$

which is a contradiction. ∎

To continue, we define a particular class of partitions:

Definition 3.3. For each $n \in \mathbb{N}$,

$$\mathscr{A}_n = \bigvee_{j=0}^{n} T^{-j}\mathscr{A}_0$$

It is immediately clear that T^{n+1} is monotone and smooth on each element of the partition $\mathscr{A}_n$ (in fact, this could be used as an alternative definition of $\mathscr{A}_n$); moreover, $\mathscr{A}_n$ consists of intervals with Lebesgue measure smaller than λ^{-n}.

We can then choose n_0 such that all the elements of $\mathscr{A}_{n_0}$ have measure less than $1/(2a)$ (for example $n_0 = [\ln 2a/\ln \lambda] + 1$ would do).

Definition 3.4. We call a map "covering" if for each $n \in \mathbb{N}$ there exists $N(n)$ such that, for each $I \in \mathscr{A}_n$,[14]

$$T^{N(n)}I = [0, 1]$$

The above property corresponds to condition (H2) in ref. 6. The importance of the notion of "covering" is emphasized by the following lemma.

Lemma 3.5. If the map T is covering, then for each $a > A/(1-2\lambda^{-1})$ there exists $\Delta > 0$ such that

$$\operatorname{diam}(\tilde{T}^{N(n_0)}\mathscr{C}_a) \leqslant \Delta < \infty$$

Proof. Let $g \in \mathscr{C}_a$; then, according to Lemma 3.2, there exists $I_0 \in \mathscr{A}_{n_0}$ such that $g(x) \geqslant \frac{1}{2}\int_0^1 g$ for each $x \in I_0$. By the covering property, for each

[14] We have already remarked that what happens at the boundaries of the partitions is immaterial. In the same vein, each time that we write an equality between sets we always mean it apart from a finite number of points.

$x \in [0, 1]$ (apart from at most finitely many points) there exists $y \in I_0$ such that $T^{N(n_0)}y = x$; hence

$$(\tilde{T}^{N(n_0)}g)(x) = \sum_{y \in T^{-N(n_0)}x} g(y) \, |D_y T^{N(n_0)}|^{-1} \geqslant \frac{\int_0^1 g}{2 \, \|DT\|_\infty^{N(n_0)}}$$

Lemma 2.1 implies $\tilde{T}^{N(n_0)}\mathscr{C}_a \subset \mathscr{C}_{\sigma_1, a}$ with[15]

$$\sigma_1 = (2\lambda^{-1})^{N(n_0)} + \frac{1 - (2\lambda^{-1})^{N(n_0)}}{1 - 2\lambda^{-1}} A a^{-1}$$

Let $\delta(g) = (\inf \tilde{T}^{N(n_0)} g)/\int g$; up to now we have seen that $\delta(g) \geqslant (2 \, \|DT\|^{N(n_0)})^{-1}$ for all $g \in \mathscr{C}_a$. Using Lemma 3.1, we can then estimate

$$\begin{aligned}
\operatorname{diam}(\tilde{T}^{N(n_0)}\mathscr{C}_a) &\leqslant 2 \sup_{g \in \mathscr{C}_a} \ln \left[\frac{\max\{(1+\sigma_1)\int g; \inf \tilde{T}^{N(n_0)} g + \bigvee_0^1 \tilde{T}^{N(n_0)} g\}}{\min\{(1-\sigma_1)\int g; \inf \tilde{T}^{N(n_0)} g\}} \right] \\
&\leqslant 2 \sup_{g \in \mathscr{C}_a} \ln \left[\frac{\max\{(1+\sigma_1); \delta(g) + a\sigma_1\}}{\min\{(1-\sigma_1); \delta(g)\}} \right] \\
&\leqslant 2 \ln \left[\frac{\max\{(1+\sigma_1); 1 + a\sigma_1\}}{\min\{(1-\sigma_1); (2 \, \|DT\|_\infty^{N(n_0)})^{-1}\}} \right] \equiv \Delta \quad \blacksquare
\end{aligned}$$

The above lemma, together with the results of Section 1, implies exponential decay of the correlations for covering maps.

Theorem 3.6. If T is covering, then for each $f \in L^1([0, 1])$ and $g \in BV$, $\int_0^1 g = 1$,

$$\left| \int_0^1 g f \circ T^n - \int_0^1 f \, d\mu \right| \leqslant K_n \Lambda^n \, \|f\|_1 \left(1 + b \bigvee_0^1 g \right) \qquad \forall n \in \mathbb{N}$$

with

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right)^{1/N(n_0)}$$

[15] To obtain the following formula, it is enough to notice that, iterating (2.1), for each $k > 0$, we have

$$\bigvee_0^1 \tilde{T}^k g \leqslant (2\lambda^{-1})^k \bigvee_0^1 g + \frac{1 - (2\lambda^{-1})^k}{1 - 2\lambda^{-1}} A \int_0^1 g$$

Apply then Lemma 2.1 directly to $\tilde{T}^{N(n_0)}$.

$$K_n = \{\exp[\Delta \Lambda^{(n-N(n_0))}]\} \Lambda^{-N(n_0)\Delta} \|\phi\|_\infty$$
$$b = (a-B)^{-1}$$

$[B = A/(1-2\lambda^{-1})]$.[16]

Proof. Choose $a > A/(1-2\lambda^{-1})$ and consider $g \in \mathscr{C}_a$ normalized so that $\int_0^1 g = 1$ (i.e., g can be thought of as the density of a measure). Then,

$$\left|\int_0^1 f(\tilde{T}^n g - \phi)\, dx\right| \leqslant \|f\|_1 \left\|\frac{\tilde{T}^n g}{\phi} - 1\right\|_\infty \|\phi\|_\infty$$

Lemma 3.7. If $\mathscr{C}_+ = \{g \in BV \mid g(x) \geqslant 0, \forall x \in [0, 1]\}$ and Θ_+ is the corresponding Hilbert metric, then for each $g_1, g_2 \in \mathscr{C}_a$,

$$\Theta_+(g_1, g_2) \leqslant \Theta(g_1, g_2)$$

Proof. Since $\mathscr{C}_+ \supset \mathscr{C}_a$, the identity is a map from BV to itself that maps $\mathscr{C}_a$ into $\mathscr{C}_+$. The result follows then from Theorem 1.1. ▮

A simple computation yields

$$\Theta_+(g_1, g_2) = \ln \sup_{x, y \in [0, 1]} \frac{g_1(x)\, g_2(y)}{g_1(y)\, g_2(x)}$$

Using the previous facts and the trivial equality

$$\frac{(\tilde{T}^n g)(x)}{\phi(x)} = \frac{\tilde{T}^n g(x)\, \phi(y)}{\tilde{T}^n g(y)\, \phi(x)} \frac{\tilde{T}^n g(y)}{\phi(y)}$$

we have

$$\exp[-\Theta_+(\tilde{T}^n g, \phi)] \frac{\tilde{T}^n g(y)}{\phi(y)} \leqslant \frac{\tilde{T}^n g(x)}{\phi(x)} \leqslant \exp[\Theta_+(\tilde{T}^n g, \phi)] \frac{\tilde{T}^n g(y)}{\phi(y)}$$

for each $x, y \in [0, 1]$. Because $\int_0^1 (\tilde{T}^n g - \phi) = 0$, there must exist $y_n^+, y_n^- \in [0, 1]$ such that $\tilde{T}^n g(y_n^-) \leqslant \phi(y_n^-)$ and $\tilde{T}^n g(y_n^+) \geqslant \phi(y_n^+)$. Using the previous inequalities with $y = y_n^-$ and $y = y_n^+$, respectively, we obtain, for each $x \in [0, 1]$,

$$\exp[-\Theta_+(\tilde{T}^n g, \phi)] \leqslant \frac{\tilde{T}^n g(x)}{\phi(x)} \leqslant \exp[\Theta_+(\tilde{T}^n g, \phi)]$$

[16] Note that, since $\Lambda < 1$, $\lim_{n\to\infty} K_n = \Lambda^{-N(n_0)\Delta} \|\phi\|_\infty \leqslant \Lambda^{-N(n_0)\Delta}(1+a)$.

and

$$\left\| \frac{\tilde{T}^n g}{\phi} - 1 \right\|_\infty \leqslant \exp[\Theta_+(\tilde{T}^n g, \phi)] - 1 \leqslant \exp[\Theta(\tilde{T}^n g, \phi)] - 1$$

According to Theorem 1.1 and Lemma 3.5,

$$\begin{aligned}\Theta(\tilde{T}^n g, \phi) &\leqslant \Theta([\tilde{T}^{N(n_0)}]^{[n/N(n_0)]} g, [\tilde{T}^{N(n_0)}]^{[n/N(n_0)]} \phi) \\ &\leqslant \tanh\left(\frac{\Delta}{4}\right)^{[n/N(n_0)]-1} \Theta(\tilde{T}^{N(n_0)} g, \phi)\end{aligned}$$

Hence,

$$\begin{aligned}\left\| \frac{\tilde{T}^n g}{\phi} - 1 \right\|_\infty &\leqslant \exp[\Lambda^{n-N(n_0)} \Theta(\tilde{T}^{N(n_0)} g, \phi)] - 1 \\ &\leqslant \{\exp[\Lambda^{n-N(n_0)} \Delta]\} \, \Delta \Lambda^{n-N(n_0)}\end{aligned}$$

This estimate shows that for each $f \in L^1([0, 1])$, $g \in \mathscr{C}_a$, $n \in \mathbb{N}$,

$$\left| \int_0^1 f \circ T^n g \, dx - \int_0^1 f\phi \, dx \right| \leqslant K_n \, \|f\|_1 \, \Lambda^n$$

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right)^{1/N(n_0)}; \qquad K_n = \{\exp[\Lambda^{n-N(n_0)} \Delta]\} \, \Delta \Lambda^{-N(n_0)} \, \|\phi\|_\infty$$

Let us now consider $g \in BV$, $g \geqslant 0$, and $\int_0^1 g = 1$. If $\bigvee_0^1 g \leqslant a$, we have the above estimate; otherwise we define $g_\rho = (g + \rho\phi)(1+\rho)^{-1}$; then $\int_0^1 g_\rho = 1$ and

$$\bigvee_0^1 g_\rho = \left[\bigvee_0^1 g + \rho \bigvee_0^1 \phi \right] (1+\rho)^{-1}$$

Iterating (2.1), one obtains $\bigvee_0^1 \phi \leqslant A/(1 - 2\lambda^{-1}) = B$; then

$$\bigvee_0^1 g_\rho \leqslant \left[\bigvee_0^1 g + \rho B \right] (1+\rho)^{-1}$$

Choosing

$$\rho = \frac{\bigvee_0^1 g - a}{a - B}$$

we have $g_\rho \in \mathscr{C}_a$; hence, the decay of correlations for g_ρ implies the decay of correlations for g. The result for arbitrary $g \in BV$ follows, since any function can be written as the difference of two positive functions. ∎

In the next section we will see that, provided $\inf \phi > 0$, all the mixing maps are covering, thereby completing the proof of Theorem 0.1.

4. GENERAL PROPERTIES OF EXPANDING MAPS

In this section we address some questions concerning piecewise expanding maps posed by Collet.[6] We will see that the property that some image of any interval covers all $[0, 1]$ is a quite general feature of mixing maps. This shows that the results obtained in this paper apply to a wide class of maps.

We start by giving a verifiable criterion for the hypotheses of Theorem 0.1.

Definition 4.1. We call a map "weakly covering" if there exists $N_0 \in \mathbb{N}$ such that, for each $I \in \mathscr{A}_0$,

$$\bigcup_{j=0}^{N_0} T^j I = [0, 1]$$

This is a weaker version of (H1) in ref. 6. The next lemma shows that weak covering is all that is needed to ensure that the first of the hypotheses of Theorem 0.1 holds.

Lemma 4.2. If a map is weakly covering, then there exists $\gamma > 0$ such that $\inf \phi \geqslant \gamma$.

Proof. A consequence of weak covering and expansivity is that the property of being weakly covering does not depend substantially on the partition.

Sublemma 4.3. If a map is weakly covering, then, for each $n \in \mathbb{N}$, there exists $N_0(n) \in \mathbb{N}$ such that, for each $I \in \mathscr{A}_n$,

$$\bigcup_{j=0}^{N_0(n)} T^j I = [0, 1]$$

Proof. Let $I \in \mathscr{A}_n$; then T^n is smooth on I; accordingly, $|T^n I| \geqslant \lambda^n |I|$. If $T^n I$ covers an element of $\mathscr{A}_0$, then the lemma is proven; if not, since $T^n I$ is connected, it can intersect at most two elements of $\mathscr{A}_0$, so it is naturally broken in at most two pieces; let I_1 be the larger of the two; clearly, $|I_1| \geqslant \lambda^n |I|/2$.

We can then carry out a recursive argument: consider TI_1; by construction it is connected; intersect it with the elements of $\mathscr{A}_0$; either it will cover one element or it will be divided in at most two subintervals; call I_2 the larger one; consider TI_2, and so on. It follows that $|I_k| \geqslant (\lambda/2)^{k-1} |I_1|$, which implies that eventually I_k will cover an element of $\mathscr{A}_0$; this is all that is needed to prove the lemma. ∎

By definition, $\tilde{T}\phi = \phi$; in addition, by Lemma 3.2 there exists $I_0 \in \mathscr{A}_{n_0}$ such that $\phi(x) \geqslant 1/2$ for each $x \in I_0$. By Sublemma 4.3, for each $x \in [0, 1]$ there exists $j \leqslant N_0(n_0)$ and $y_* \in I_0$ such that $T^j y_* = x$. Hence,

$$\phi(x) = \tilde{T}^j \phi(x) = \sum_{y \in T^{-j}x} \phi(y) \, |D_y T^j|^{-1} \geqslant \phi(y_*) \, \|DT\|_\infty^{-j} \geqslant \tfrac{1}{2} \, \|DT\|_\infty^{-N_0(n_0)} \quad ∎$$

The main result of this section is contained in the following theorem.

Theorem 4.4. If an expanding map is mixing and $\inf \phi \geqslant \gamma > 0$, then it is covering.

Proof. For each interval $I \subset [0, 1]$, define

$$\chi_I(x) = \begin{cases} 1/|I| & \text{for} \quad x \in I \\ 0 & \text{for} \quad x \notin I \end{cases}$$

Then, $\bigvee_0^1 \chi_I \leqslant 2/|I|$ and $\int_0^1 \chi_I = 1$. Given any two intervals I, I', the mixing property implies

$$\lim_{n \to \infty} \int_0^1 \chi_{I'} \chi_I \circ T^n = \int_0^1 \chi_I \phi \geqslant \gamma$$

Consider some $n_1 \in \mathbb{N}$ (to be chosen later); there exists $N_* \in \mathbb{N}$ such that, for each $I, I' \in \mathscr{A}_{n_1}$,

$$\int_0^1 \chi_I \tilde{T}^n \chi_{I'} \geqslant \frac{\gamma}{2} \qquad \forall n \geqslant N_*$$

Choose $I_0 \in \mathscr{A}_{n_1}$; consider $\tilde{T}^n \chi_{I_0}$; from (2.1) it follows that

$$\bigvee_0^1 \tilde{T}^n \chi_{I_0} \leqslant (2\lambda^{-1})^n \frac{2}{|I_0|} + A \sum_{i=0}^{n-1} (2\lambda^{-1})^i \int_0^1 \chi_{I_0} \leqslant (2\lambda^{-1})^n \frac{2}{|I_0|} + \frac{A}{1 - 2\lambda^{-1}}$$

Let $b_n = \inf_{I \in \mathscr{A}_n} |I|$; choose $N_1 \geqslant N_*$ such that

$$\frac{2(2\lambda^{-1})^{N_1}}{b_{n_1}} \leqslant \frac{A}{1 - 2\lambda^{-1}}$$

Then,

$$\bigvee_0^1 \tilde{T}^n \chi_{I_0} \leqslant \frac{2A}{1-2\lambda^{-1}} \qquad \forall n \geqslant N_1$$

Consider the set $\mathscr{B}_- = \{I \in \mathscr{A}_{n_1} \mid \text{exists } x \in I\colon \tilde{T}^{N_1}\chi_{I_0}(x) < \gamma/4\}$, and let $L = \#\mathscr{B}_-$ ($\#$ means the cardinality of a set).

First of all, for each $I \in \mathscr{A}_{n_1}$ there exists $y \in I$ such that $\tilde{T}^{N_1}\chi_{I_0}(y) \geqslant \gamma/2$, if not,

$$\int_0^1 \chi_I \tilde{T}^{N_1} \chi_{I_0} < \frac{\gamma}{2} \int_0^1 \chi_I = \frac{\gamma}{2}$$

contrary to our assumptions on N_1.

Consequently, for each $I \in \mathscr{B}_-$,

$$\bigvee_I \tilde{T}^{N_1} \chi_{I_0} \geqslant \frac{\gamma}{4}$$

which implies

$$L \leqslant \frac{8A}{(1-2\lambda^{-1})\gamma} \equiv L_0$$

But, if $\#\{T^{-n}x\} \leqslant L_0$, we have

$$\gamma \leqslant \phi(x) = \sum_{y \in T^{-n}x} \phi(y) \, |D_y T^n|^{-1} \leqslant L_0 \, \|\phi\|_\infty \, \lambda^{-n}$$

which shows that, for $N_2 = [\ln L_0 \, \|\phi\|_\infty \, \gamma^{-1}/\ln \lambda] + 1$, $\#\{T^{-N_2}x\} > L_0$.

Choose $n_1 = N_2$; since $T^{-n_1}x$ has at most one point in each element of $\mathscr{A}_{n_1}$ and $\#\{T^{-n_1}x\} > L_0$, it follows that there exists $y \in T^{-n_1}x$ such that $y \in I \notin \mathscr{B}_-$; hence $T^{n_1+N_1}I_0 = [0, 1]$.

The statement is then proven by using the same reasoning employed in Sublemma 4.3. ∎

Summarizing, if a map is weakly covering and mixing, then, in view of Lemma 4.2, we have $\gamma > 0$ and, by Theorem 4.4, the map is covering. Hence, we can prove the exponential decay of correlations thanks to Theorem 3.6. We have proved Theorem 0.1.

APPENDIX A. THE CONSTANTS AND AN EXAMPLE

The constants Λ, K, and b in Theorem 0.1 can be chosen as

$$\Lambda = \left[\frac{1-e^{-\Delta/2}}{1+e^{-\Delta/2}}\right]^{1/N^*} < 1$$

$$K = e^{\Lambda^{-N^*}}\Lambda^{-N^*}\Delta(1+a)$$

$$b = \frac{1-2\lambda^{-1}}{A}$$

Several quantities here must be defined: $\lambda = \inf_{\xi \in [0,1]} |D_\xi T|$ is the minimum expansion rate;

$$A = \sup_{\xi \in [0,1]} \frac{|D_\xi^2 T|}{|D_\xi T|^2} + 2 \sup_{I \in \mathscr{A}_0} \frac{\sup_{\xi \in I} |D_\xi T|^{-1}}{|I|}$$

is the constant that appears in the main inequality (2.1);

$$\Delta = 2 \ln\{[3 + (2\lambda^{-1})^{N^*}](\sup_{\xi \in [0,1]} |D_\xi T|)^{N^*}\}$$

is an estimate of the diameter of the image of the cone. In addition,

$$a = \max\left\{1, \frac{2A}{1-2\lambda^{-1}}\right\}$$

is the parameter that fixes the choice of the cone.[17]

The last quantity we need to define is N^*. Unfortunately, the choice of N^* is not so simple. Recall that $\mathscr{A}_n$ is the coarser partition, in intervals, of $[0, 1]$ such that T^{n+1} is C^2 on each interval; then, for each $n \in \mathbb{N}$, the number $N(n)$ is defined as the smallest integer for which

$$T^{N(n)}I = [0, 1] \qquad \forall I \in \mathscr{A}_n$$

We choose $n_0 = [\ln 2a/\ln \lambda] + 1$ and define

$$N^* = \max\left\{N(n_0), \left[\frac{\ln 2}{\ln \lambda - \ln 2}\right] + 1\right\}$$

In Section 4 we give abstract conditions ensuring that, for each $n \in \mathbb{N}$, $N(n)$ exists finite, but I do not know any general bounds on $N(n)$, so, in a concrete example, one must construct the partition $\mathscr{A}_{n_0}$ and iterate its elements to find the value of N^*.

[17] The reader may have noticed that a is never explicitly chosen in the body of the paper. In fact, all the estimates obtained depend on the choice of a; therefore the best strategy would be to chose a last by optimizing the bounds. Such an approach would lead to very complicated formulas; to avoid this here I make some choice, neither the best nor the worst.

The above formulas are not the sharpest general bounds that can be obtained from the results contained in this paper: they represent a compromise between reasonable bounds and reasonable formulas. Nevertheless, it is important to remark that given a specific example, it is possible to improve the estimates by following step by step the construction carried out in Section 3.

To be more concrete, let us look at an example.

Example. Consider the piecewise linear map

$$T(x)=\begin{cases} \frac{9}{2}(\frac{1}{9}-x) & x\in(0,\frac{1}{9})\\ \frac{9}{2}(x-\frac{1}{9}) & x\in(\frac{1}{9},\frac{3}{9})\\ \frac{9}{2}(\frac{5}{9}-x) & x\in(\frac{3}{9},\frac{5}{9})\\ \frac{9}{2}(x-\frac{5}{9}) & x\in(\frac{5}{9},\frac{7}{9})\\ \frac{9}{2}(1-x) & x\in(\frac{7}{9},1)\end{cases}$$

The partition on which T is defined is

$$\mathscr{A}_0=\{(0,\tfrac{1}{9});(\tfrac{1}{9},\tfrac{1}{3});(\tfrac{1}{3},\tfrac{5}{9});(\tfrac{5}{9},\tfrac{7}{9});(\tfrac{7}{9},1)\}$$

The map satisfies our assumptions since $|DT|=9/2>2$ and it is easy to check that it is covering (remember that, by Lemma 4.2 and Theorem 4.4, it suffices to check that T is weakly covering).

The smaller element of the partition $\mathscr{A}_0$ has size 1/9; accordingly, $A=4$ [see (2.1)]. The choice of a is subject to the constraint $a>A/(1-2\lambda^{-1})=36/5$ (see Lemma 2.1), while n_0 must satisfy $\sup_{I\in\mathscr{A}_{n_0}}|I|\leqslant 1/(2a)$; it is then immediately clear that we must choose, at least, $n_0=1$.

The partition $\mathscr{A}_1$ is

$$\mathscr{A}_1=\{(0,\tfrac{1}{27});(\tfrac{1}{27},\tfrac{7}{81});(\tfrac{7}{81},\tfrac{1}{9});(\tfrac{1}{9},\tfrac{11}{81});(\tfrac{11}{81},\tfrac{15}{81});\ldots\}$$

Clearly $\inf_{I\in\mathscr{A}_1}|I|=2/81$.

All the elements of $\mathscr{A}_1$ have as image an element of $\mathscr{A}_0$ that it is mapped on all [0, 1] apart for the ones that are mapped on (0, 1/9) and for (0, 1/27). A direct computation shows that $T^3(0,1/27)=[0,1]$ and $T^2(0,1/9)=[0,1]$, so $N(1)=3$.

If we choose $a=7.25$, then $\sigma_1\leqslant 0.994$ (see Lemma 3.5 for a definition of σ_1).[18] Since $a\sigma_1\geqslant 1+\sigma_1$ and $1-\sigma_1\geqslant\frac{1}{2}(\frac{2}{9})^3$, the formula in the proof of Lemma 3.5 yields

$$\frac{\Delta}{2}\leqslant\ln[1+2a\sigma_1\,\|DT\|^{N(1)}]\leqslant\ln 1314$$

[18] Note that the general formulas at the beginning of the section would lead to the choice $a=72/5$ and $n_0=3$. Since $N(3)=5$, it would follow that $\Lambda\leqslant 1-1/14000$; we will see that a more careful approach can improve the estimate by almost two orders of magnitude.

$$\Lambda = \tanh\left(\frac{\Delta}{4}\right)^{1/3} \approx 1 - \frac{2}{3} e^{-\Delta/2} \leqslant 1 - \frac{1}{1971}$$

Moreover, $\lim_{n \to \infty} K_n \leqslant 140$ and $b \leqslant 0.3$.

The above estimates are probably off by a couple of orders of magnitude, but they should not be considered too unsatisfactory: using the only other known rigorous estimates,[6] one would have been led to consider at least $N(20)$ obtaining an estimate of $(1-\Lambda)^{-1}$ wrong by at least 20 orders of magnitude. In addition, the present estimate can certainly be improved: first of all it is easy to check that for each $x \in [0, 1]$, we have $\#\{T^{-3}x\} \geqslant 2$; this allows us immediately to divide by two the number inside the log (cf. the proof of Lemma 3.5); second, some advantages can be obtained by using a different partition (instead of the dynamical one) and a different number (instead of 1/2) in Lemma 3.2.[19] Moreover, the result can conceivably be improved by choosing a cone of functions better adapted to the particular example at hand.

However, the problem of finding optimal bounds it is a business all in itself and not the one we have been concerned with in this paper.

APPENDIX B. THE MAIN INEQUALITY

Given $I \in \mathscr{A}_0$, $I = [a(I), b(I)]$, and letting χ_I be the characteristic function of I, we have

$$\begin{aligned}
\bigvee_0^1 \tilde{T} g \chi_I &= \bigvee_I g\, |DT|^{-1} + |g(a(I))|\, |D_{a(I)} T|^{-1} + |g(b(I))|\, |D_{b(I)} T|^{-1} \\
&\leqslant \bigvee_I g\, |DT|^{-1} + \sup_{\xi \in I} |D_\xi T|^{-1} \left(\bigvee_I g + 2 \inf_{\xi \in I} |g(\xi)| \right) \\
&\leqslant 2\lambda^{-1} \bigvee_I g + \left(\sup_{\xi \in I} \frac{|D_\xi^2 T|}{|D_\xi T|^2} + \frac{2}{|I|} \sup_{\xi \in I} |D_\xi T|^{-1} \right) \int_I g
\end{aligned}$$

Since $\bigvee_0^1 \tilde{T} g \leqslant \sum_{I \in \mathscr{A}_0} \bigvee_0^1 \tilde{T} g \chi_I$, inequality (2.1) follows.

[19] Using these two ideas (i.e., the partition $\{(0, 1/9); (1/9, 2/9); (2/9, 3/9);\ldots\}$ and 7/36 instead of 1/2, which allows us to set $n_0 = 0$), it is already possible to obtain the improved estimate $\Lambda \leqslant 1 - 1/377$.

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REFERENCES

1. V. Baladi and G. Keller, Zeta functions and transfer operators for piecewise monotone transformations, *Commun. Math. Phys.* **127**:459–477 (1990).
2. M. Benedicks and L.-S. Young, Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps, *Ergodic Theory Dynam. Syst.* **12**:13–37 (1992).
3. Garret Birkhoff, Extension of Jentzsch's theorem, *Trans. Am. Math. Soc.* **85**:219–227 (1957).
4. Garret Birkhoff, *Lattice Theory*, 3rd ed. (American Mathematical Society Colloquium Publications, Providence, Rhode Island, 1967).
5. P. Diaconis and D. Stroock, Geometric bounds for eigenvalues of Markov chains, *Ann. Appl. Prob.* **1**:36–61 (1991).
6. P. Collet, An estimate of the decay of correlations for mixing non Markov expanding maps of the interval, preprint (1994).
7. P. Ferrero, Contribution à la théorie des états d'équilibre en méchanique statistique, Theses (1981).
8. P. Ferrero and B. Schmitt, Produits aléatories d'opérateurs matrices de transfert, *Prob. Theory Related Fields* **79**:227–248 (1988).
9. P. Ferrero and B. Schmitt, On the rate of convergence for some limit ratio theorem related to endomorphisms with a nonregular invariant density, preprint.
10. P. Ferrero and B. Schmitt, Ruelle's Perron–Frobenius theorem and projective metrics, in *Random Fields, Esztergom (Hungary), Colloquia Mathematica Societas János Bolyai*, Vol. 27 (1979).
11. F. Hofbauer and G. Keller, Ergodic properties of invariant measures for piecewise monotone transformations, *Math. Z.* **180**:119–140 (1982).
12. B. R. Hunt, Estimating invariant measures and Lyapunov exponents, preprint.
13. G. Keller, Un théorème de la limite centrale pour une classe de transformations monotones par morceaux, *C. R. Acad. Sci. A* **291**:155–158 (1980).
14. C. Ionescu-Tulcea and Marinescu, *Ann. Math.* **52**:140–147 (1950).
15. C. Liverani, Decay of correlations, to appear in *Ann. Math.*
16. A. Lasota and J. Yorke, On existence of invariant measures for piecewise monotonic transformations, *Trans. Am. Math. Soc.* **186**:481–487 (1973).
17. D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, New York, 1978).
18. D. Ruelle, The thermodynamic formalism for expanding maps, *Commun. Math. Phys.* **125**:239–262 (1989).
19. D. Ruelle, An extension of the theory of fredholm determinants, *IHES* **72**:175–193 (1990).
20. Marek Rychlik, Regularity of the metric entropy for expanding maps, *Trans. Am. Math. Soc.* **315**(2):833–847 (1989).
21. H. Samelson, On the Perron–Frobenius theorem, *Michigan Math. J.* **4**:57–59 (1956).
22. L.-S. Young, Decay of correlations for certain quadratic maps, *Commun. Math. Phys.* **146**:123–138 (1992).
23. K. Ziemian, On the Perron–Frobenius operator and limit theorems for some maps of an interval, *Ergodic Theory and Related Topics II, Proceedings* (Teubner, 1987), pp. 206–211.